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# SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY (Cohomology Theory of Finite Groups and Related Topics)

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# SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY

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Given a finite group  $G$  of automorphisms of a compact Riemann surface, we discuss a relation between Mumford-Morita-Miller classes of odd indices and the homology representation of  $G$ . Since most participants were group theorists rather than topologists, I separate the algebraic and the topological ingredients and explain the former in detail.

## 1. SURFACE SYMMETRIES

**1.1. The Grieder group of a finite group.** Let  $G$  be a finite group and  $\hat{\gamma}$  the conjugacy class of  $\gamma \in G$ . We denote by  $\langle \hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_q \rangle$  an unordered  $q$ -tuple ( $q \geq 0$ ) of conjugacy classes of nontrivial elements of  $G$  satisfying  $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$ , and  $\mathcal{M}_G$  the set of all such  $q$ -tuples. We can define an abelian monoid structure on  $\mathcal{M}_G$  by

$$\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle + \langle \hat{\gamma}_{q+1}, \dots, \hat{\gamma}_r \rangle = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q, \hat{\gamma}_{q+1}, \dots, \hat{\gamma}_r \rangle.$$

The identity element is the empty tuple  $\langle \rangle$ . We call  $\mathcal{M}_G$  the *Grieder monoid* of  $G$ . Now let  $\mathcal{M}'_G$  be the submonoid generated by  $\langle \hat{\gamma}, \hat{\gamma}^{-1} \rangle$  ( $\gamma \in G$ ) and set  $\mathcal{A}_G := \mathcal{M}_G / \mathcal{M}'_G$ . The quotient  $\mathcal{A}_G$  is an abelian group. The inverse element is given by

$$-\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle = \langle \hat{\gamma}_1^{-1}, \dots, \hat{\gamma}_q^{-1} \rangle \text{ in } \mathcal{A}_G.$$

We call  $\mathcal{A}_G$  the *Grieder group* of  $G$ . As the names suggest,  $\mathcal{M}_G$  and  $\mathcal{A}_G$  were introduced and studied by Grieder [5, 6] to study surface symmetries. First of all,  $\mathcal{A}_G$  is finitely generated:

**Proposition 1 ([5]).**  $\mathcal{A}_G \cong \mathbb{Z}^m \oplus \mathbb{Z}_2^n$  for some  $m, n \geq 0$ .

A homomorphism  $f: H \rightarrow G$  of groups induces a homomorphism  $f_*: \mathcal{A}_H \rightarrow \mathcal{A}_G$  of abelian groups by  $f_*(\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle) = \langle f(\hat{\gamma}_1), \dots, f(\hat{\gamma}_q) \rangle$  so that the assignment  $G \mapsto \mathcal{A}_G$  is a covariant functor. In addition, for an inclusion  $i: H \hookrightarrow G$ , one can also define the restriction  $i^*: \mathcal{A}_G \rightarrow \mathcal{A}_H$  via surface symmetries. Grieder [5] verified the double coset formula and hence proved the following proposition:

**Proposition 2.** *The assignment  $G \mapsto \mathcal{M}_G$  is a Mackey functor.*

**1.2. Ramification data.** By a *surface symmetry* we mean a pair  $(G, C)$ , where  $C$  is a compact Riemann surface of genus  $g \geq 2$ , and  $G$  is a finite group of automorphisms of  $C$ . For each  $x \in C$ , let  $G_x$  be the isotropy subgroup at  $x$ . Note that  $G_x$  is necessarily cyclic. Set  $S = \{x \in C \mid G_x \neq 1\}$ , and let  $S/G = \{x_1, x_2, \dots, x_q\}$  be a set of representatives of  $G$ -orbits of elements of  $S$ . For each  $x_i \in S/G$ , choose a generator  $\gamma_i$  of  $G_{x_i}$  such that  $\gamma_i$  acts on the holomorphic tangent space  $T_{x_i}C$  by  $z \mapsto \exp(2\pi\sqrt{-1}/|G_{x_i}|)z$  with respect to a suitable local coordinate  $z$  at  $x_i$ . The *ramification data* of  $(G, C)$ , abbreviated by  $\delta(G, C)$ , is the unordered  $q$ -tuple  $\langle \hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_q \rangle$ . It satisfies  $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$ , and hence  $\delta(G, C)$  is an element of the Grieder monoid  $\mathcal{M}_G$ . Conversely, we have the following proposition.

**Proposition 3** (see [5]). *For any element  $\mu \in \mathcal{M}_G$ , there exists a surface symmetry  $(G, C)$  whose ramification data coincides with  $\mu$ .*

## 2. GROUP COHOMOLOGY

**2.1. The first Chern class.** Let  $\langle \gamma \rangle$  be a cyclic group of order  $m$  generated by  $\gamma$  and  $\rho_\gamma : \langle \gamma \rangle \rightarrow \mathbb{C}^\times$  a linear character defined by  $\gamma \mapsto \exp(2\pi i/m)$ . For any finite group  $G$ , we have natural isomorphisms

$$\mathrm{Hom}(G, \mathbb{C}^\times) \cong H^1(G, \mathbb{C}^\times) \cong H^2(G, \mathbb{Z}).$$

Here, the latter isomorphism is the connecting homomorphism associated to the short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^\times \rightarrow 0$ . Define  $c(\gamma) \in H^2(\langle \gamma \rangle, \mathbb{Z})$  to be the image of  $\rho_\gamma$  under the isomorphism  $\mathrm{Hom}(\langle \gamma \rangle, \mathbb{C}^\times) \cong H^2(\langle \gamma \rangle, \mathbb{Z})$ . The cohomology class  $c(\gamma)$  is sometimes called the *first Chern class* of  $\rho_\gamma$ .

**2.2. MMM classes (algebra).** For each element  $\mu = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle$  of  $\mathcal{M}_G$ , define a series of cohomology classes  $e_k(\mu) \in H^{2k}(G, \mathbb{Z})$  ( $k \geq 1$ ) by

$$e_k(\mu) := \sum_{i=1}^q \mathrm{Tr}_{\langle \gamma_i \rangle}^G (c(\gamma_i)^k) \in H^{2k}(G, \mathbb{Z}),$$

where  $\mathrm{Tr}_{\langle \gamma \rangle}^G : H^*(\langle \gamma \rangle, \mathbb{Z}) \rightarrow H^*(G, \mathbb{Z})$  is the transfer. We call  $e_k(\mu)$  the *k-th Mumford-Morita-Miller class* of  $\mu$  (MMM class in short). The definition of  $e_k(\mu)$  is motivated by topology, as will be explained in the next subsection. Observe that the assignment  $\mu \mapsto e_k(\mu)$  defines a well-defined homomorphism  $\mathcal{M}_G \rightarrow H^{2k}(G, \mathbb{Z})$  of abelian monoids. For  $k$  odd, it induces a well-defined homomorphism  $\mathcal{A}_G \rightarrow H^{2k}(G, \mathbb{Z})$  of abelian groups, for we have  $c(\gamma^{-1}) = -c(\gamma)$ . In addition, we can prove the following proposition:

**Proposition 4.** For odd  $k \geq 1$ , the homomorphism  $\mathcal{A}_G \rightarrow H^{2k}(G, \mathbb{Z})$  is a natural transformation of Mackey functors.

**2.3. MMM classes (topology).** The definition of  $e_k(\mu)$  is inspired by a result of Kawazumi and Uemura [8] concerning of characteristic classes of oriented surface bundles. Let  $\Sigma_g$  be the closed oriented surface of genus  $g \geq 2$ . Let  $\pi : E \rightarrow B$  an oriented  $\Sigma_g$ -bundle,  $T^\vee E$  the tangent bundle along the fiber of  $\pi$ , and  $e \in H^2(E; \mathbb{Z})$  the Euler class of  $T^\vee E$ . Define  $e_k^{\text{top}}(\pi) \in H^{2k}(B; \mathbb{Z})$  by  $e_k^{\text{top}}(\pi) := \pi_!(e^{k+1})$  where  $\pi_! : H^*(E; \mathbb{Z}) \rightarrow H^{*-2}(B; \mathbb{Z})$  is the Gysin homomorphism (the superscript “top” stands for “topology”).  $e_k^{\text{top}}(\pi)$  is called the  $k$ -th Mumford-Morita-Miller class of  $\pi$ , as it was introduced in [11, 10, 9].

Now let  $(G, C)$  be a surface symmetry as in Section 1.2. Associated with  $(G, C)$ , there is an oriented surface bundle  $\pi : EG \times_G C \rightarrow BG$  called the *Borel construction*, where  $EG \rightarrow BG$  is the universal  $G$ -bundle. We denote by  $e_k^{\text{top}}(G, C) \in H^{2k}(G, \mathbb{Z})$  the  $k$ -th MMM class of the Borel construction  $\pi$ . A result of Kawazumi and Uemura [8] implies the following result:

**Theorem 5.** We have  $e_k^{\text{top}}(G, C) = e_k(\delta(G, C))$  where  $\delta(G, C)$  is the ramification data of  $(G, C)$ .

### 3. HOMOLOGY REPRESENTATIONS

**3.1. Algebra.** In what follows, we denote by  $R(G)$  the complex representation ring (or the character ring) of a finite group  $G$ . Let  $\langle \gamma \rangle$  be a cyclic group of order  $m$  generated by  $\gamma$  and  $\rho_\gamma : \langle \gamma \rangle \rightarrow \mathbb{C}^\times$  a linear character as in Section 2.1. Define  $\Delta_\gamma \in R(\langle \gamma \rangle) \otimes \mathbb{Q}$  by

$$\Delta_\gamma := 2 \sum_{k=1}^{m-1} \left( \frac{k}{m} - \frac{1}{2} \right) \rho_\gamma^{\otimes k} = \frac{2}{m} \sum_{k=1}^{m-1} k \rho_\gamma^{\otimes k} - r_{\langle \gamma \rangle} + 1_{\langle \gamma \rangle},$$

where  $r_{\langle \gamma \rangle}$  is the regular representation and  $1_{\langle \gamma \rangle}$  is the trivial 1-dimensional representation of  $\langle \gamma \rangle$ . Now, for each element  $\mu = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle$  of  $\mathcal{M}_G$ , define the  $G$ -signature  $\sigma(\mu)$  of  $\mu$  by

$$\sigma(\mu) := \sum_{k=1}^q \text{Ind}_{\langle \gamma_k \rangle}^G (\Delta_{\gamma_k}) \in R(G) \otimes \mathbb{Q}.$$

**Proposition 6.**  $\sigma(\mu) \in R(G)$  for every  $G$  and  $\mu \in \mathcal{M}_G$ .

See the next section for the proof. Note that, in case  $\mu \in \mathcal{M}_G$  consists of a single conjugacy class ( $\mu = \langle \hat{\gamma} \rangle$  for  $\gamma \in [G, G]$ ), Proposition 6 was proved by T. Yoshida [13].

The assignment  $\mu \mapsto \sigma(\mu)$  yields a homomorphism  $\mathcal{M}_G \rightarrow R(G)$  of monoids, which induces a well-defined homomorphism  $\mathcal{A}_G \rightarrow R(G)$  of abelian groups. In addition, we can prove the following proposition:

**Proposition 7.**  $\mathcal{A}_G \rightarrow R(G)$  is a natural transformation of Mackey functors.

**3.2. Topology.** Let  $(G, C)$  be a surface symmetry, and  $H_C$  the space of holomorphic 1-forms on  $C$ . Note that  $\dim_{\mathbb{C}} H_C = g$  where  $g$  is the genus of the Riemann surface  $C$ . Then  $G$  acts on  $H_C$  and hence  $H_C$  is a complex representation of  $G$ . A virtual representation  $\sigma^{\text{top}}(G, C) := H_C - \overline{H}_C \in R(G)$  is called the  $G$ -signature of  $(G, C)$ , where  $\overline{H}_C$  is the complex conjugate.

**Proposition 8.** We have  $\sigma^{\text{top}}(G, C) = \sigma(\delta(G, C))$  where  $\delta(G, C)$  is the ramification data of  $(G, C)$ .

The character of  $\sigma^{\text{top}}(G, C)$  is given by the Eichler trace formula (see [4] for instance). The proposition can be verified by comparing characters of  $\sigma^{\text{top}}(G, C)$  and  $\sigma(\delta(G, C))$ . An alternative proof was given by N. Kawazumi (unpublished manuscript). Since every  $\mu \in \mathcal{M}_G$  can be realized as a ramification data of a surface symmetry, Proposition 6 follows from the last proposition. The following fact is an easy consequence of Proposition 8.

**Corollary 9.** If all the complex characters of  $G$  are  $\mathbb{R}$ -valued, then  $\sigma(\mu) = 0$  for all  $\mu \in \mathcal{M}_G$ .

*Proof.* Choose a surface symmetry  $(G, C)$  with  $\delta(G, C) = \mu$ . Then we have  $\sigma(\mu) = \sigma^{\text{top}}(G, C) = H_C - \overline{H}_C = 0$  since  $H_C = \overline{H}_C$  by the assumption.  $\square$

#### 4. A RELATION OF $e_k(\mu)$ AND $\sigma(\mu)$

**Theorem 10.** Let  $G$  be a finite group and  $\mu, \nu \in \mathcal{M}_G$ .

- (1) If  $\sigma(\mu) = \sigma(\nu)$  then  $e_k(\mu) = e_k(\nu)$  for all odd  $k \geq 1$ .
- (2) If  $\sigma(\mu) = 0$  then  $e_k(\mu) = 0$  for all odd  $k \geq 1$ .

Since  $R(G)$  is free as an abelian group, the homomorphism  $\mathcal{A}_G \rightarrow R(G)$  in Section 3.1 induces  $\phi_1 : \mathcal{A}_G / \text{Tor}(\mathcal{A}_G) \rightarrow R(G)$ , where  $\text{Tor}(\mathcal{A}_G)$  is the torsion subgroup of  $\mathcal{A}_G$ . For odd  $k \geq 1$ , let  $\phi_2 : \text{Tor}(\mathcal{A}_G) \rightarrow H^{2k}(G, \mathbb{Z})$  be the restriction of the homomorphism  $\mathcal{A}_G \rightarrow H^{2k}(G, \mathbb{Z})$  in Section 2.2. The proof of Theorem 10 is based on the following two facts:

**Theorem 11.** For any finite group  $G$  and any odd  $k \geq 1$ ,

- (1) The homomorphism  $\phi_1 : \mathcal{A}_G / \text{Tor}(\mathcal{A}_G) \rightarrow R(G)$  is injective.
- (2) The homomorphism  $\phi_2 : \text{Tor}(\mathcal{A}_G) \rightarrow H^{2k}(G, \mathbb{Z})$  is trivial.

The first statement is proved by using a result of Edmonds and Ewing [3], while the second statement is proved by considering the cohomology of metacyclic 2-groups. The detail will appear elsewhere. Theorem 10 and Corollary 9 imply the following corollary:

**Corollary 12.** *If all the complex characters of  $G$  are  $\mathbb{R}$ -valued, then  $e_k(\mu) = 0$  for all  $\mu \in \mathcal{M}_G$  and odd  $k \geq 1$ .*

Define  $\mathcal{R}_G$  to be the image of  $\phi_1 : \mathcal{A}_G / \text{Tor}(\mathcal{A}_G) \rightarrow R(G)$ . In view of Theorem 11, there exists a series of homomorphisms  $\Phi_k : \mathcal{R}_G \rightarrow H^{2k}(G, \mathbb{Z})$  ( $k$  odd) which assigns  $e_k(\mu)$  to  $\sigma(\mu)$ . Let  $c : \text{Hom}(G, \mathbb{C}^\times) \rightarrow H^2(G, \mathbb{Z})$  be the natural isomorphism as in Section 2.1 and  $\det : R(G) \rightarrow \text{Hom}(G, \mathbb{C}^\times)$  the determinant homomorphism (see [13] for precise). Then the homomorphism  $\Phi_1$  is determined by the following proposition:

**Proposition 13.**  $e_1(\mu) = 6 \cdot c(\det(\sigma(\mu)))$  for all  $\mu \in \mathcal{M}_G$ .

The proposition follows from the Grothendieck-Riemann-Roch theorem and a result of Harer [7] (see also [1, Proposition 6]). Proposition 13 can be generalized to larger  $k$ , provided  $G$  is cyclic. Recall that, for every finite group  $G$ , there is a series of homomorphisms  $s_k : R(G) \rightarrow H^{2k}(G, \mathbb{Z})$  ( $k \geq 0$ ) of abelian groups, which satisfies the following properties:

- (1)  $s_1(\rho) = c(\det \rho)$  for all  $\rho \in R(G)$ .
- (2) If  $\rho$  is a linear character, then  $s_k(\rho) = c(\rho)^k$ .

$s_k(\rho)$  is called the  $k$ -th Newton class of  $\rho \in R(G)$ . See [12] for further details. Let  $B_{2k}$  be the  $2k$ -th Bernoulli number and  $N_{2k}, D_{2k}$  coprime integers satisfying  $B_{2k}/k = N_{2k}/D_{2k}$ . Then a result of the author and Kawazumi [2] implies the following result:

**Theorem 14.** *If  $G$  is cyclic, then  $N_{2k} \cdot e_{2k-1}(\mu) = D_{2k} \cdot s_{2k-1}(\sigma(\mu))$  holds for all  $\mu \in \mathcal{M}_G$  and  $k \geq 1$ .*

Now let  $G$  be a cyclic group of order  $m$ , and suppose that  $N_{2k}$  is prime to  $m$ . Choose an integer  $N_{2k}^*$  satisfying  $N_{2k} \cdot N_{2k}^* \equiv 1 \pmod{m}$ . Under these assumptions, we have

$$e_{2k-1}(\mu) = N_{2k}^* D_{2k} \cdot s_{2k-1}(\sigma(\mu))$$

for all  $\mu \in \mathcal{M}_G$ , and hence determining  $\Phi_{2k-1}$  for these cases. In particular, we have  $e_1(\mu) = 6 \cdot s_1(\sigma(\mu))$ ,  $e_3(\mu) = -60 \cdot s_3(\sigma(\mu))$ ,  $e_5(\mu) = 126 \cdot s_5(\sigma(\mu))$ ,  $e_7(\mu) = -120 \cdot s_7(\sigma(\mu))$  for any cyclic group  $G$  and  $\mu \in \mathcal{M}_G$ , since  $N_{2k} = 1$  for  $1 \leq k \leq 4$ .

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